

On 1-Harmonic Functions^{*}

Shihshu Walter WEI

Department of Mathematics, The University of Oklahoma, Norman, Ok 73019-0315, USA

E-mail: wwei@ou.edu

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Abstract. Characterizations of entire subsolutions for the 1-harmonic equation of a constant 1-tension field are given with applications in geometry via transformation group theory. In particular, we prove that every level hypersurface of such a subsolution is calibrated and hence is area-minimizing over \mathbb{R} ; and every 7-dimensional $SO(2) \times SO(6)$ -invariant absolutely area-minimizing integral current in \mathbb{R}^8 is real analytic. The assumption on the $SO(2) \times SO(6)$ -invariance cannot be removed, due to the first counter-example in \mathbb{R}^8 , proved by Bombieri, De Giorgi and Giusti.

Key words: 1-harmonic function; 1-tension field; absolutely area-minimizing integral current

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1 Introduction

The study of 1-harmonic functions, or more generally that of p -harmonic maps is an area of an active research that is related with many branches of mathematics. For instance, in a celebrated paper of Bombieri, De Giorgi and Giusti [3], a 1-harmonic function has been constructed to provide a counter-example for interior regularity of the solution to the co-dimension one Plateau problem in \mathbb{R}^n for $n > 7$. Recall a C^1 functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be 1-harmonic if it is a weak solution of 1-harmonic equation

$$\operatorname{div} \left(\frac{\nabla f}{|\nabla f|} \right) = 0, \quad (1.1)$$

where $|\nabla f|$ is the length of the gradient ∇f of f , and for a C^2 function f without a critical point, $\operatorname{div} \left(\frac{\nabla f}{|\nabla f|} \right)$ is said to be the 1-tension field of f .

In this paper, characterizations of entire subsolutions for the 1-harmonic equation of a constant 1-tension field are given in various aspects, and their relationships with calibration geometry are established (cf. Theorem 2, Corollary 3). As applications, we prove via transformation group theory (cf. [9, 10, 13, 2, 21]) that the cone over $S^1 \times S^5$ is not minimizing in \mathbb{R}^8 but is stable; that any 7-dimensional $SO(2) \times SO(6)$ -invariant absolutely area-minimizing integral current in \mathbb{R}^8 is real analytic; and that the only 7-dimensional $SO(3) \times SO(5)$ -invariant minimizing integral current with singularities in \mathbb{R}^8 is the cone over $S^2 \times S^4$, and is minimizing over \mathbb{R} (cf. Theorems 3–5). These results improved an early partial proof by numerical computation done by Plinio Simoes [17] in his Berkeley thesis. The assumption on the $SO(2) \times SO(6)$ -invariance cannot be removed, due to the first counter-example of Bombieri, De Giorgi and Giusti that the cone over $S^3(\frac{1}{\sqrt{2}}) \times S^3(\frac{1}{\sqrt{2}}) \subset S^7(1)$ is area-minimizing in \mathbb{R}^8 . It should be pointed out that Fang-Hua Lin [14] proved that the cone over $S^1 \times S^5$ is one-sided area-minimizing and is stable by a different method. By constructing 1-harmonic functions on hyperbolic space H^n , $H^n \times H^n$,

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$H^n \times SO(n, 1)$ and many other associated spaces, S.P. Wang and the author [19] show the Bernstein Conjecture in these spaces to be false in all dimensions. In particular, these constructions give the *first* set of examples of complete, smooth, embedded, minimal (hyper-)surfaces in hyperbolic space H^n in all dimensions (cf. also Remark 3(ii)).

2 Fundamentals in geometric measure theory

For our subsequent development, we recall some fundamental facts, definitions, and notations, for which the reference is Federer's book [5] and paper [7].

Let N denote an n -dimensional Riemannian manifold and denote by $\mathcal{R}_p^{\text{loc}}(N)$ the set of p -dimensional, locally rectifiable currents (of Federer and Fleming, cf. [8]) on N . For $S \in \mathcal{R}_p^{\text{loc}}(N)$, denote the mass of S by $\mathbf{M}(S)$, and the *boundary* of S by ∂S , and is given by $(\partial S)(w) = S(dw)$, where w is a smooth p -form and d is the exterior differentiation. From a calculus of variational viewpoint, we make the following

Definition 1. A current $T \in \mathcal{R}_k^{\text{loc}}(N)$ is said to be *stationary* if $\frac{d}{dt}\mathbf{M}(\phi_{t*}^V(T))|_{t=0}$ for all vector fields V on N with compact support where ϕ_t^V is the flow associated with V , and *stable* if for every vector fields V on N with compact support, there exists an $\epsilon > 0$ such that $\mathbf{M}(T) \leq \mathbf{M}(\phi_{t*}^V(T))$ for $|t| < \epsilon$.

We are primarily interested in minimizing currents.

Definition 2. A current $T \in \mathcal{R}_k^{\text{loc}}(N)$ is *homologically* (resp. *absolutely*) *area-minimizing* over \mathbb{Z} if for all compact sets $K \subset M$, we have $\mathbf{M}(\phi_K T) \leq \mathbf{M}((\phi_K T) + S)$ for all $S \in \mathcal{R}_k^{\text{loc}}(N)$ having compact support and being the boundary of some current in $\mathcal{R}_{k+1}^{\text{loc}}(N)$ with compact support (resp. the empty boundary) (here ϕ_K denotes the characteristic function on K).

Using a dimension reduction technique, Federer proves that the support of an area-minimizing integral current T [8] minus another compact set S whose Hausdorff dimension does not exceed $n - 8$ is an $(n - 1)$ -dimensional analytic manifold [6]. Hence, if $n \leq 7$, then $S = \emptyset$. If $n = 8$, S consists of at most isolated points [5, 5.4.16]. This result is optimal by the counter-example due to Bombieri–De Giorgi–Giusti [3] that $\{x \in \mathbb{R}^{2m} : x_1^2 + \cdots + x_m^2 = x_{m+1}^2 + \cdots + x_{2m}^2\}$ is an area-minimizing cone over the product of $(m - 1)$ -spheres $\{x \in \mathbb{R}^{2m} : x_1^2 + \cdots + x_m^2 = x_{m+1}^2 + \cdots + x_{2m}^2 = \frac{1}{2}\}$ in \mathbb{R}^{2m} for $m \geq 4$.

The union of the groups $\mathcal{F}_{m,K}(U) = \{R + \partial T : R \in \mathcal{R}_{m,K}(U), T \in \mathcal{R}_{m+1,K}(U)\}$ corresponding to all compact $K \subset U$ is the group $\mathcal{F}_m(U)$ of m -dimensional *integral flat chains in an open subset* U of \mathbb{R}^n . We denote the group of m -dimensional *integral flat chains, cycles and boundaries* by $\mathcal{F}_m(A) = \mathcal{F}_m(\mathbb{R}^n) \cap \{S : \text{spt } S \subset A\}$, $\mathcal{Z}_m(A, B) = \mathcal{F}_m(A) \cap \{S : \partial S \subset \mathcal{F}_m(B) \text{ or } m = 0\}$, and $\mathcal{B}_m(A, B) = \{R + \partial T : R \in \mathcal{F}_m(B), T \in \mathcal{F}_{m+1}(A)\}$ respectively. Similarly, we define and denote $\mathbf{F}_m(A)$, $\mathbf{Z}_m(A, B)$ and $\mathbf{B}_m(A, B)$ the vector space of m -dimensional *real flat chains, cycles and boundaries* respectively, where $B \subset A$ are compact Lipschitz neighborhood retract in U .

For every positive convex parametric integrand ψ , and every compact subset K of A , we define $\mathcal{Z}_{m,K}(A, B) = \mathcal{Z}_m(A, B) \cap \{R : \text{spt } R \subset K\}$, $\mathcal{B}_{m,K}(A, B) = \mathcal{B}_m(A, B) \cap \{R : \text{spt } R \subset K\}$, $\mathbf{Z}_{m,K}(A, B) = \mathbf{Z}_m(A, B) \cap \{R : \text{spt } R \subset K\}$, and $\mathbf{B}_{m,K}(A, B) = \mathbf{B}_m(A, B) \cap \{R : \text{spt } R \subset K\}$, and make the following

Definition 3. An m -dimensional rectifiable current Q (resp. Q') is said to be *absolutely* (resp. *homologically*) ψ -*minimizing in* K *with respect to* (A, B) *over* \mathbb{Z} if

$$\int_Q \psi = \inf \left\{ \int_S \psi : S \in \mathcal{F}_{m,K}(U), Q - S \in \mathcal{Z}_{m,K}(A, B) \right\} \\ \left(\text{resp. } \int_{Q'} \psi = \inf \left\{ \int_S \psi : S \in \mathcal{B}_{m,K}(U), Q' - S \in \mathcal{B}_{m,K}(A, B) \right\} \right).$$

Definition 4. An m -dimensional real flat chain Q (resp. Q') is said to be *absolutely* (resp. *homologically*) ψ -minimizing in K with respect to (A, B) over \mathbb{R} if

$$\int_Q \psi = \inf \left\{ \int_S \psi : S \in \mathbf{F}_{m,K}(U), Q - S \in \mathbf{Z}_{m,K}(A, B) \right\} \\ \left(\text{resp. } \int_{Q'} \psi = \inf \left\{ \int_S \psi : S \in \mathbf{B}_{m,K}(U), Q' - S \in \mathbf{B}_{m,K}(A, B) \right\} \right).$$

We will make comparisons between real and integral absolute (resp. homological) minimizing currents in the subsequent Sections 3, 4, and 5.

3 Characterizations of subsolutions for 1-harmonic equation of constant 1-tension field

We connect an entire subsolution of this sort, with a calibration. Recall a calibration is a closed form with comass 1.

Lemma 1. *Let M be a complete noncompact Riemannian manifold. For any $x_0 \in M$ and any pair of positive numbers s, t with $s < t$, there exists a rotationally symmetric Lipschitz continuous function $\psi(x) = \psi(x; s, t)$ and a constant $C_1 > 0$ (independent of x_0, s, t) with the properties:*

$$(i) \quad \psi \equiv 1 \text{ on } B(x_0; s), \text{ and } \psi \equiv 0 \text{ off } B(x_0; t); \\ (ii) \quad |\nabla \psi| \leq \frac{C_1}{t-s}, \text{ a.e. on } M. \quad (3.1)$$

Proof. (cf. Andreotti and Vesentini [1], Yau [22], Karp [11]). ■

Theorem 1. *Let Ω be a domain in \mathbb{R}^n containing a ball $B(x_0, r)$ of radius r , centered at x_0 , and $g : \Omega \rightarrow \mathbb{R}$ be a continuous function with $g \geq 0$, and $c = \inf_{x \in B(x_0, \frac{r}{2})} g(x)$. Let $f : \Omega \rightarrow \mathbb{R}$ be a C^1 weak solution of*

$$\operatorname{div} \left(\frac{\nabla f}{|\nabla f|} \right) = g(x) \quad \text{on } \Omega, \quad (3.2)$$

then the infimum c satisfies

$$0 \leq c \leq \frac{C_1 2^n}{r},$$

where C_1 is as in (3.1).

Proof. Let $\psi \geq 0$ be as in Lemma 1, in which $M = \mathbb{R}^n$, $t = r$, $s = \frac{r}{2}$. Choose ψ to be a test function in the distribution sense of (3.2). Then via the assumption on g , and Cauchy-Schwarz inequality we have:

$$\int_{B(x_0, \frac{r}{2})} c\psi(x)dx \leq \int_{B(x_0, \frac{r}{2})} g(x)\psi(x)dx \\ \leq \int_{B(x_0, r)} g(x)\psi(x)dx = - \int_{B(x_0, r)} \frac{\nabla f}{|\nabla f|} \cdot \nabla \psi dx \leq \int_{B(x_0, r)} |\nabla \psi| dx.$$

Hence,

$$c \operatorname{Vol} \left(B \left(x_0, \frac{r}{2} \right) \right) \leq \frac{C_1}{r} \operatorname{Vol}(B(x_0, r))$$

yields the desired. ■

Corollary 1. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a C^1 weak subsolution of 1-harmonic equation (1.1) with constant 1-tension field c , i.e. $0 \leq \operatorname{div} \left(\frac{\nabla f}{|\nabla f|} \right) = c$ in the distribution sense. Then f is a 1-harmonic function.*

Corollary 2. *There does not exist a C^1 weak subsolution $f : \mathbb{R}^n \rightarrow \mathbb{R}$ of equation (3.2) with $\lim_{r \rightarrow \infty} \inf_{x \in B(x_0, r)} g(x) > 0$, for any $x_0 \in \mathbb{R}^n$.*

Let $A \subset \mathbb{R}^n$ be an open set. We denote $BV_{\text{loc}}(A) = \{f \in L^1_{\text{loc}}(A) : \text{the distributional derivatives } D_i f \text{ of } f \text{ are (locally) measures}\} = \{f \in L^1_{\text{loc}}(A) : \text{supp } \phi_n \subset K \subset A, \phi_n \rightarrow 0 \text{ uniformly, imply } \left(\frac{\partial}{\partial x_i} f \right) \phi_n \rightarrow 0\}$. Let $Df = (D_1 f, \dots, D_n f)$ denote the gradient of f in the sense of distributions and $|Df|$ the scalar measure defined by $\int_K |Df| = \sup \int_K \sum_i \epsilon_i(x) D_i f$, where the supremum is taken over all sets $\{\epsilon_i(x), i = 1, \dots, n\}$ of $C^\infty(K)$ functions which satisfy $\sum \epsilon_i^2(x) \leq 1$.

Definition 5. A function $f \in BV_{\text{loc}}(A)$ has least gradient in A if for every $g \in BV_{\text{loc}}(A)$, with compact support $K \subset A$ we have

$$\int_K |Df| \leq \int_K |D(f + g)|. \quad (3.3)$$

Definition 6. Let E be a set in \mathbb{R}^n and ϕ_E its characteristic function. E has an oriented boundary of least area with respect to A , if (i) $\phi_E \in BV_{\text{loc}}(A)$ and (ii) for each $g \in BV_{\text{loc}}(A)$ with compact support $K \subset A$ we have $\int_K |D\phi_E| \leq \int_K |D(\phi_E + g)|$.

Theorem 2. *Let $f \in H^{1,1}_{\text{loc}}(\mathbb{R}^n)$, and $\nabla f(x) \neq 0$ for every x in \mathbb{R}^n . Let $E_\lambda = \{x : f(x) \geq \lambda\}$, and $S_\lambda = \{x : f(x) = \lambda\}$. We denote the set of integers by \mathbb{Z} . Then the following thirteen statements (1)–(13) are equivalent and each of them implies the fourteenth statement (14).*

1. $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a C^1 weak subsolution of (1.1) with constant 1-tension field.
2. f is a C^1 weak solution of (1.1) on \mathbb{R}^n .
3. f is a C^1 1-harmonic function on \mathbb{R}^n .
4. For each $(a, t_0) = (a_1, \dots, a_{n-1}, t_0) \in S_\lambda$, there exists a neighborhood \mathcal{D} of a in \mathbb{R}^{n-1} , and a unique real analytic function $\eta : \mathcal{D} \rightarrow \mathbb{R}$ such that $\eta(a) = t_0$, $f(x_1, \dots, x_{n-1}, \eta(x_1, \dots, x_{n-1})) = \lambda$ and $\operatorname{div} \left(\frac{\nabla \eta}{\sqrt{1 + |\nabla \eta|^2}} \right) = 0$ on \mathcal{D} .
5. Each level hypersurface S_λ is minimal in \mathbb{R}^n .
6. $\frac{*df}{|df|}$ is a globally defined “weakly” closed form with comass 1.
7. f is a function of least gradient in \mathbb{R}^n .
8. Each E_λ , $\lambda \in \mathbb{R}$ has an oriented boundary of least area with respect to \mathbb{R}^n .
9. Each level hypersurface S_λ is absolutely area-minimizing in \mathbb{R}^n over \mathbb{Z} .
10. Each level hypersurface S_λ is absolutely area-minimizing in \mathbb{R}^n over \mathbb{R} .
11. Each level hypersurface S_λ is homologically area-minimizing in \mathbb{R}^n over \mathbb{R} .
12. Each level hypersurface S_λ is homologically area-minimizing in \mathbb{R}^n over \mathbb{Z} .
13. Each level hypersurface S_λ is stable in \mathbb{R}^n .
14. If $f \in C^2(\mathbb{R}^n)$, then $\frac{*df}{|df|}$ is closed and the restriction $\frac{*df}{|df|} \Big|_{S_\lambda}$ is its volume form, hence each S_λ is real absolutely area-minimizing in \mathbb{R}^n over \mathbb{R} .

Corollary 3. *Every level hypersurface of a C^2 subsolution of 1-harmonic equation on \mathbb{R}^{n+1} with constant 1-tension field is calibrated and hence is area-minimizing over \mathbb{R} .*

Proof. (1) \Leftrightarrow (2) \Leftrightarrow (3) : This follows immediately from Corollary 1.

(2) \Leftrightarrow (4) : (\Rightarrow) Let $f(x_1, \dots, x_{n-1}, t) = \eta(x_1, \dots, x_{n-1}) - t$. The assertion follows from the implicit function theorem and

$$0 = \int \frac{\sum_{i=1}^{n-1} \frac{\partial f}{\partial x_i} \frac{\partial \varphi}{\partial x_i}}{|\nabla f|} + \int \frac{\frac{\partial f}{\partial t} \frac{\partial \varphi}{\partial t}}{|\nabla f|} = \int \sum_{i=1}^{n-1} \frac{\frac{\partial \eta}{\partial x_i}}{\sqrt{1 + |\nabla \eta|^2}} \frac{\partial \varphi}{\partial x_i} \quad (3.4)$$

for all $\varphi \in C_0^\infty(\mathcal{D} \times \mathbb{R})$. The regularity of solutions of minimal surface equation implies that η is real analytic and completes the proof. (\Leftarrow) This follows immediately from (3.3).

(4) \Leftrightarrow (5) : This is due to the fact that the graph of a solution to the minimal surface equation on \mathcal{D} is a minimal hypersurface in $\mathcal{D} \times \mathbb{R}$.

(2) \Leftrightarrow (6) : This follows from the following: For every $\phi \in C_0^\infty(A)$,

$$\begin{aligned} \int_A \frac{*df}{|df|} \wedge d\phi &= \int_A \sum_{i,j=1}^n (-1)^{i-1} \frac{\frac{\partial f}{\partial x_i}}{|\nabla f|} dx^1 \wedge \dots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \dots \wedge dx^n \wedge \frac{\partial \phi}{\partial x_j} dx^j \\ &= \int_A \sum_{i=1}^n (-1)^{n-1} \frac{\frac{\partial f}{\partial x_i} \frac{\partial \phi}{\partial x_i}}{|\nabla f|} dx^1 \wedge \dots \wedge dx^i \wedge \dots \wedge dx^n. \end{aligned}$$

(2) \Rightarrow (7): let us first assume that $g \in C_0^1(A)$. Let $h(t) = \int |D(f + tg)|$. Then

$$h'(t) = \int \frac{\left(\sum_{i=1}^n \frac{\partial(f+tg)}{\partial x_i} \frac{\partial g}{\partial x_i} \right)}{\left(\sum_{i=1}^n \left(\frac{\partial(f+tg)}{\partial x_i} \right)^2 \right)^{\frac{1}{2}}}.$$

Hence $h'(0) = 0$ by assumption. Furthermore,

$$h''(t) = \int \frac{\left(\sum_{i=1}^n \left(\frac{\partial g}{\partial x_i} \right)^2 \right) \left(\sum_{i=1}^n \left(\frac{\partial(f+tg)}{\partial x_i} \right)^2 \right) - \left(\sum_{i=1}^n \frac{\partial(f+tg)}{\partial x_i} \frac{\partial g}{\partial x_i} \right)^2}{\left[\sum_{i=1}^n \left(\frac{\partial(f+tg)}{\partial x_i} \right)^2 \right]^{\frac{3}{2}}} \geq 0,$$

by the Cauchy-Schwarz inequality. Therefore $\int |Df| = h(0) \leq h(1) = \int |D(f + g)|$. If $g \in BV_{\text{loc}}(A)$ with compact support K and let $Dg = G_1 + G_2$ where G_1 is completely continuous and G_2 is the singular part of Dg with support N_g of measure zero. Then we have $\int_K |D(f + g)| = \int_K |Df + G_1| + \int_K |G_2|$ because $f \in H_{\text{loc}}^{1,1}(A)$. Let $g_\epsilon = g * \psi_\epsilon$ where ψ_ϵ is a mollifier. Then $g \in C_0^1(A)$ and $\int_{K_\epsilon} |Df| \leq \int_{K_\epsilon} |D(f + g_\epsilon)| \leq \int_{K_\epsilon} |Df + G_1 * \Psi_\epsilon| + \int_A |G_2 * \Psi_\epsilon|$, where $K_\epsilon = \{x \in A : \text{dist}(x, K) < \epsilon\}$. Letting $\epsilon \rightarrow 0$ completes the proof (cf. [3]).

(7) \Rightarrow (8) : This follows from Coarea formula for BV functions [15], $\int_K |Df| = \int_{-\infty}^{\infty} (\int_K |D\phi_\lambda|) d\lambda$

together with two observations:

- (i) If f_1 and f_2 satisfy (3.3), so does $\sup(f_1, f_2)$.
- (ii) If $f_i \in BV_{\text{loc}}(A)$, $f_i \rightarrow f$ in L_{loc}^1 and each f_i satisfies (3.3), so does also $f \in BV_{\text{loc}}(A)$ and satisfies (3.3).

For detailed proof see [16].

(8) \Rightarrow (9) : Let $\phi_\lambda = \phi_{E_\lambda}$. Since for every x in \mathbb{R}^n , $\nabla f(x) \neq 0$, $\partial E_\lambda = S_\lambda$ for $S_\lambda \neq \emptyset$. It follows from a theorem of Miranda [15] that on any compact set K in \mathbb{R}^n , the Hausdorff $(n-1)$ -measure

$$\mathcal{H}^{n-1}(K \cap S_\lambda) = \int_K |D\phi_\lambda| \leq \int_K |D(\phi_\lambda + g)| = \mathcal{H}^{n-1}(K \cap T)$$

for all sets T with $\partial(K \cap T) = \partial(K \cap S_\lambda)$.

(9) \Rightarrow (10) : It follows from Theorem 6.

(10) \Rightarrow (11) \Rightarrow (12) : Since absolute area-minimization over $\mathbb{R} \Rightarrow$ homological area-minimization over $\mathbb{R} \Rightarrow$ homological area-minimization over \mathbb{Z} .

(12) \Rightarrow (13) \Rightarrow (5) : Since homological minimization over $\mathbb{Z} \Rightarrow$ stability \Rightarrow minimality. This completes the proof of (1) $\Leftrightarrow \dots \Leftrightarrow$ (13).

(2) \Rightarrow (14) : If $f \in C^2(A)$ then by (3.4) $\frac{*df}{|df|}$ is closed. Now let e_1, \dots, e_{n-1} be an orthonormal basis for the tangent space of S_λ at x_0 and ν a unit normal vector at x_0 . We denote by tilde “ \sim ” the canonical isomorphism between a tangent space and its dual space. To show $\frac{*df}{|df|}$ has comass 1, note for any $(n-1)$ -vector field ξ ,

$$\begin{aligned} \frac{*df}{|df|}(\xi) &= \left(* \frac{\widetilde{\nabla f}}{|\nabla f|} \right)(\xi) \quad \left(\text{because } \frac{df}{|df|}(X) = \frac{Xf}{|\nabla f|} = \left\langle \frac{\nabla f}{|\nabla f|}, X \right\rangle \right) \\ &= (*\tilde{\nu})(\xi) = (e_1 \wedge \dots \wedge e_{n-1})(\xi) = \langle e_1 \wedge \dots \wedge e_{n-1}, \xi \rangle. \end{aligned}$$

In particular $\frac{*df}{|df|}(e_1 \wedge \dots \wedge e_{n-1}) = 1$, $\frac{*df}{|df|}(\xi) \leq 1$ and $\frac{*df}{|df|}|_{S_\lambda} = \text{volume element of } S_\lambda$. By the formalism of Stokes theorem, for any integral current T with $\partial T = \partial(S_\lambda \cap B_r)$

$$\begin{aligned} M(S_\lambda \cap B_r) &= (S_\lambda \cap B_r) \left(\frac{*df}{|df|} \right) = T \left(\frac{*df}{|df|} \right) \\ &= \int \frac{*df}{|df|}(\vec{T}_x) d\|T\|(x) \leq \int d\|T\| = M(T), \end{aligned}$$

where \vec{T} is the field of oriented unit tangent planes to T . ■

Remark 1. In Theorem 2, if one replace \mathbb{R}^n with an open subset A in \mathbb{R}^n , then assertions (2) $\Leftrightarrow \dots \Leftrightarrow$ (13) \Rightarrow (14) remain to be true.

Remark 2. Concerning the assertion (2) \Rightarrow (7), a stronger theorem can be found in [3]: Let $A \subset \mathbb{R}^n$ be an open set and let $f \in H_{\text{loc}}^{1,1}(A)$. Suppose that (i) $\mathcal{H}_n(\{x \in A : |\nabla f| = 0\}) = 0$, (ii) $\mathcal{H}_{n-1}(N) = 0$ where N is a closed set in A , (iii) $\int_{A-N} |\nabla f|^{-1} \sum_{i=1}^n \frac{\partial f}{\partial x_i} \frac{\partial \phi}{\partial x_i} dx = 0$ for every $\phi \in C_0^1(A - N)$. Then f has least gradient with respect to A .

Remark 3. (i) The assertion (7) \Rightarrow (9) is due to Miranda.

(ii) Connecting the assertions (5), (6), and (12) on Riemannian manifolds, S.P. Wang and the author [19] prove that if each level hypersurface of a smooth function $f : M \rightarrow \mathbb{R}$ on an oriented Riemannian manifold M with nowhere vanishing ∇f , is minimal, then there exists a closed form with comass 1 on M and hence each level hypersurface is homologically area-minimizing over \mathbb{R} .

Corollary 4. Let A be an open subset in \mathbb{R}^n , N be a closed subset in A with $\mathcal{H}_{n-1}(N) = 0$. Then the graph of any weak solution of the minimal surface equation $\sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\frac{\frac{\partial f}{\partial x_i}}{\sqrt{1+|\nabla f|^2}} \right) = 0$ on $A - N$ is in fact absolutely area-minimizing in $A \times \mathbb{R} \subset \mathbb{R}^{n+1}$ over \mathbb{R} .

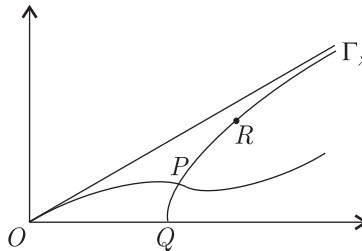
Proof. Applying (3.4) in which “ $f(x_1, \dots, x_{n-1}, t) = \eta(x_1, \dots, x_{n-1}) - t$ ” is replaced with “ $F(x_1, \dots, x_n, t) = f(x_1, \dots, x_n) - t$ ”, and Remark 2, we have that F is a C^1 1-harmonic function in A . By Theorem 2, the zero level set $S_0 = \{(x_1, \dots, x_n, t) : t = f(x_1, \dots, x_n)\}$ is absolutely area-minimizing in $A \times \mathbb{R} \subset \mathbb{R}^{n+1}$ over \mathbb{R} . ■

4 Further applications

A natural question arises: Are Bombieri–De Giorgi–Giusti and Lawson cones the only $SO(m) \times SO(n)$ -invariant singular absolutely area-minimizing integral currents in Euclidean space \mathbb{R}^{m+n+2} ? The answer is affirmative. Combining the theory of 1-harmonic functions developed, and the techniques of transformation groups in [10, 13, 2], and [21], evolved from the ideas in [9], one obtains the following:

Theorem 3. *The cone $C(S^m \times S^n)$ over $S^m \times S^n$ is the unique singular absolutely area-minimizing hypersurface in the class of $SO(m+1) \times SO(n+1)$ -invariant integral currents in \mathbb{R}^{m+n+2} over \mathbb{R} for $m+n > 7$ or $m+n = 6$, $|m-n| \leq 2$. (It is known that the cone is not even stable otherwise.)*

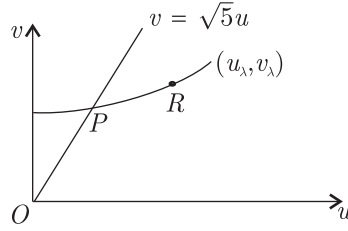
Proof. Assume $m = n$. Let Lie group $G = SO(n+1) \times SO(n+1)$ acting on manifold $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ in the standard way, i.e. assigning $((A, B), (x, y)) \in G \times \mathbb{R}^{2n+2}$ to $(A \cdot x, B \cdot y) \in \mathbb{R}^{2n+2}$, where “ \cdot ” is the matrix multiplication. Then the collection X of principle orbits is given by $X = \{(x, y) \in \mathbb{R}^{2n+2} : |x||y| \neq 0\}$, where “ $|\cdot|$ ” is the length of “ \cdot ” in \mathbb{R}^{n+1} . The orbit space which is stratified, can be represented as $\mathbb{R}^{2n+2}/G = \{(u, v) \in \mathbb{R}^2 : u, v \geq 0\} = X \cup \{(u, v) \in \mathbb{R}^2 : u = 0, v > 0\} \cup \{(u, v) \in \mathbb{R}^2 : u > 0, v = 0\} \cup \{(0, 0)\}$. The canonical metric on \mathbb{R}^{2n+2}/G (compatible with the fibration over each stratum) is the usual flat one $ds_0^2 = du^2 + dv^2$. The canonical projection $\pi : \mathbb{R}^{2n+2} \rightarrow \mathbb{R}^{2n+2}/G$ is given by $\pi(x, y) = (|x|, |y|)$, and let $X/G = \pi(X)$. Then the length of a curve σ in $(X/G, ds_0^2)$ is the length of any orthogonal trajectory through the corresponding orbits in X , and $2n$ -dimensional volume of $\pi^{-1}((u, v))$ (which is diffeomorphic to $S^n \times S^n$) is proportional to $u^n v^n$, for $(u, v) \in X/G$. Thus if we choose the metric $ds^2 = u^{2n} v^{2n} (du^2 + dv^2)$ on \mathbb{R}^{2n+2}/G , then by Fubini’s theorem, the length of a curve σ in $(\mathbb{R}^{2n+2}/G, ds^2)$ is equal to $(2n+1)$ -dimensional volume of hypersurface $\pi^{-1}\sigma$ (with possible singularities) in \mathbb{R}^{2n+2} , up to a constant factor. It follows that σ is a length minimizing geodesic “downstairs” (in $(\mathbb{R}^{2n+2}/G, ds^2)$), if and only if $\pi^{-1}\sigma$ is area-minimizing in the class of G -invariant $(2n+1)$ -dimensional currents “upstairs” (in $(\mathbb{R}^{2n+2}, dx_1^2 + \dots + dx_{2n+2}^2)$), or equivalently, $\pi^{-1}\sigma$ is area-minimizing in $(\mathbb{R}^{2n+2}, dx_1^2 + \dots + dx_{2n+2}^2)$ in general (cf. [13], [2, p. 174, 6.4] and [21]). Furthermore, if a length minimizing geodesic σ meets the boundary $\{(u, v) \in \mathbb{R}^2 : u = 0, v > 0\} \cup \{(u, v) \in \mathbb{R}^2 : u > 0, v = 0\}$, it meets the boundary orthogonally by the first variational formula for the arc-length functional, and the corresponding $\pi^{-1}\sigma$ is a regular, embedded and analytic hypersurface in \mathbb{R}^{2n+2} . If σ meets the vertex $\{(0, 0)\}$, then $\pi^{-1}\sigma$ is singular. Therefore, it suffices to show that any curve in \mathbb{R}^{2n+2}/G , other than the diagonal ray emanating from the origin is not absolutely length minimizing with respect to the metric $ds^2 = u^{2n} v^{2n} (du^2 + dv^2)$.



Now let $\Gamma = \{(u_0(t), v_0(t))\}$ be the geodesic through $(1, 0)$ in $(\mathbb{R}^{2n+2}/G, ds^2)$, and $\Gamma_\lambda = \{(\lambda u_0(t), \lambda v_0(t))\}$, $\lambda > 0$. In [3], a 1-harmonic function was constructed in such a way that the lift of family $\{\Gamma_\lambda\}$ of these homothetic geodesics are level hypersurfaces in $(\mathbb{R}^{2n+2}, dx_1^2 + \dots + dx_{2n+2}^2)$. Hence Γ_λ is absolutely length minimizing in $(\mathbb{R}^{2n+2}/G, ds^2)$ (cf. also Theorem 2, Remark 2). Now suppose Theorem 3 were not true. Then there would exist a curve $QP \subset \Gamma_\lambda$ transverse to a length minimizing curve OP . It follows that the length $l(OP)$ of OP would satisfy $l(OP) = l(QP)$. Consider the curve OPR where R is on the curve Γ_λ , and $l(OPR) = l(QPR)$. Then the curve OPR would be a geodesic, and hence smooth at P . This is a contradiction. Similarly, one can show the remaining case $m \neq n$. ■

Theorem 4. *The cone $C(S^1 \times S^5)$ over $S^1 \times S^5$ is not absolutely area-minimizing, although it is stable.*

Proof. Suppose, on the contrary, that the cone were absolutely area-minimizing. Then consider Lie group $G = SO(2) \times SO(6)$ acting on manifold $\mathbb{R}^2 \times \mathbb{R}^6$ in the standard way. By the previous argument, this would imply the line segment \overline{OP} were length-minimizing in $(\mathbb{R}^8/G, ds^2)$, where $ds^2 = u^2 v^6 (du^2 + dv^2)$. On the other hand, based on the study of Simoes' thesis [17], [13] and [21], the level curve (u_λ, v_λ) in the u, v -plane is absolutely length-minimizing. Argue as before, the curve OPR would be smooth at P . This is a contradiction. The stability of the cone follows from Simons' work [18]. ■



Theorem 5. *Any 7-dimensional $SO(2) \times SO(6)$ -invariant absolutely area-minimizing integral current in \mathbb{R}^8 is real analytic.*

Proof. By the argument given in the proof of Theorem 3, it suffices to show that any curve in \mathbb{R}^{2n+2}/G , from the origin is not absolutely length minimizing with respect to the metric $ds^2 = u^2 v^6 (du^2 + dv^2)$. By Theorem 4, the diagonal ray emanating from the origin is not length minimizing. Similarly, if there were an absolutely length minimizing curve starting from the origin lying above $v = \sqrt{5}u$, then this would lead to an irregularity of a geodesic, a contradiction. ■

5 Comparison theorem

It is known that each level hypersurface of a function of least gradient defined on an open subset $A \subset \mathbb{R}^n$ is absolutely area-minimizing in A over \mathbb{Z} . It is tempting to ask it if is absolutely area-minimizing in A over \mathbb{R} . This motivates our discussion on comparison between real and integral absolute (or homological) minima. In general they are distinct. Examples are given by Almgren [7, 5.11], Federer [7] and Lawson [12]. Furthermore, in the case of 1-dimensional (or co-dimension 1) integral flat chains, Federer [7] has shown that real and integral homological (or absolute) minimizing are the same.

Let \overline{M} be a locally Lipschitz neighborhood retract in \mathbb{R}^n (i.e. there exists a locally Lipschitz map which retracts a neighborhood of \overline{M} onto \overline{M}), M be an open subset of \overline{M} , and A be an open subset of \mathbb{R}^n . Using the assumption on vanishing topology, an exhaustion of M by an increasing sequence of compact set $K_i \subset M$, we obtain the following:

Theorem 6. (1) Let T^{n-1} denote a codimension 1 integral absolutely area-minimizing rectifiable current in M with homology group $H_{n-1}(\overline{M}) = 0$. Then T^{n-1} is absolutely area-minimizing in M if and only if T^{n-1} is absolutely area-minimizing in A ; and if and only if T^{n-1} is real absolutely area-minimizing in A . (2) Let $H_1(\overline{M}) = 0$. T^1 is a homologically area-minimizing rectifiable current of degree 1 of M if and only if T^1 is real homologically area-minimizing in M .

We have the following immediate

Corollary 5. The level hypersurface of a function of least gradient in an open subset A of \mathbb{R}^n is absolutely area-minimizing over \mathbb{R} .

Corollary 6. Let N be a closed set in $A \subset \mathbb{R}^n$ with $H_{n-1}(N) = 0$. The graph of any weak solution of the minimal surface equation $\sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\frac{\frac{\partial f}{\partial x_i}}{\sqrt{1+|\nabla f|^2}} \right) = 0$ on $A - N$ is in fact absolutely area-minimizing in $A \times \mathbb{R} \subset \mathbb{R}^{N+1}$ over \mathbb{R} .

Corollary 7. All the examples we find in [21] are absolutely area-minimizing over \mathbb{R} .

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